# Generalizing graphs of elementary 2-D curves using parametric functions

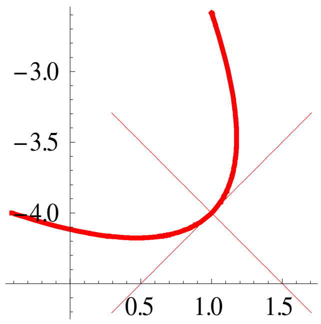
Draw the graph of $y = x^2$! In earlier lectures, I made quite a bit of a fuss about the lack of clarity hidden in this command. Let's see now if there is anything fruitful to be gained from examining the assumptions that allowed you to execute it.

The graph of $y = x^2$ in $R^2$ is a set of the form Q = {(x, y); $y = x^2$ } or {(x, $x^2$); x ∈ $R$ }. Considering Q as a subset of coordinate points on the xy-plane amounts to viewing (x, $x^2$) as a code for the point x**i** + $x^2$**j**. On the other hand, we have every mathematical right to think of Q = {(x, y); $y = x^2$ }= {(s, t); $t = s^2$ }= {(s, $s^2$); s ∈ $R$ } as a set of points on some uv-plane. In particular, the coordinate (s, $s^2$) may be viewed as a code for the point s**u** + $s^2$ **v**, where **u** and **v** are unit directional vectors of the uv-coordinate system.

To prevent miscommunications, we agree to regard every element (x, y) ∈ Q as a coordinate with respect to the horizontal x and the vertical y axes unless otherwise specified. If we wish to describe a collection of points of the form s**u** + $s^2$ **v**, we simply express the vectors **u** and **v**, as well as the origin of the uv-coordinate system in terms of their **i**, **j**, and **k** components.

**Example:** Identify the curve described parametrically by **Q** = $\left\{\left(1+\frac{s-s^2}{\sqrt{2}}, -4+\frac{s+s^2}{\sqrt{2}}\right); s \in R\right\}$

**Solution:** Observe that every element of **Q** can be written as $(1,-4)+s\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)+s^2\left(\frac{-1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ = (1, -4) + s**u** + $s^2$ **v**. This means that **Q** ≡ Q = {(s, $s^2$); s ∈ $R$ }. In more simple terms, **Q** is just a parabola in the uv-plane. The parabola and the uv-plane are drawn below.



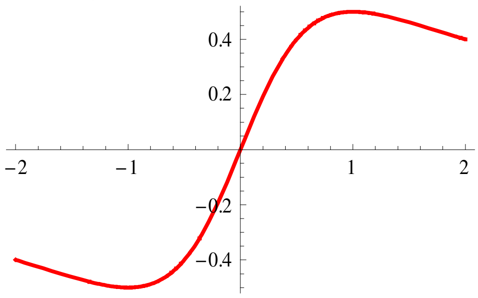

Generally, if a plane is described parametrically by the equation (x, y, z) = **p** + s**u** + t**v** ≡ (s, t), then (s, f(s)) ≡ **p** + s**u** + f(s)**v** would correspond to the drawing of the graph of v = f(u) in the uv-plane.

**Example:** Identify the curve described parametrically by

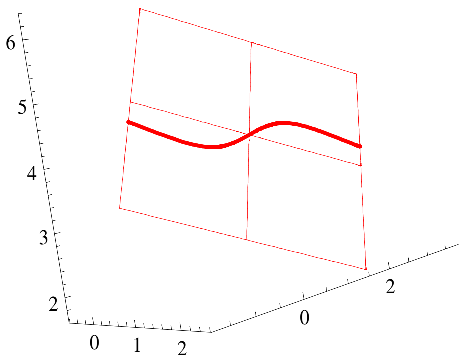
$$\left(1+\frac{s}{\sqrt{6}}-\frac{2s}{\sqrt{29}(s^2+1)}, 1+\frac{2s}{\sqrt{6}}+\frac{3s}{\sqrt{29}(s^2+1)}, 4-\frac{s}{\sqrt{6}}+\frac{4s}{\sqrt{29}(s^2+1)}\right)$$

**Solution:** We can write this as (1, 1, 4) + s $\left(\frac{1}{\sqrt{6}},\frac{2}{\sqrt{6}},\frac{-1}{\sqrt{6}}\right)$ + $\frac{s}{s^2+1}\left(\frac{-2}{\sqrt{29}},\frac{3}{\sqrt{29}},\frac{4}{\sqrt{29}}\right) \equiv$ $\left(s, \frac{s}{s^2+1}\right)$.

Thus, when we view this curve from the appropriate perspective, the curve looks like the 2-D graph described by the equation $v=\frac{u}{u^2+1}$:




With respect to the xyz-coordinate system, however, this drawing is laid on the tilted plane 11x – 2y + 7z –35 = 0:




If a curve is described by the parametric equation (x, y) = (f(s), g(s)) on the xy-plane, then **p** + f(s)**u** + g(s)**v** would have the same graph as (f(s), g(s)) relative to the uv-plane **p** + s**u** + t**v**, provided that **u** and **v** are orthogonal unit vectors. For example, (1, 1, 4) + Cos(s) (1, 2, -1) + Sin(s) (-2, 3, 4) is an ellipse with center (1, 1, 4), semimajor radius $\sqrt{29}$ and semiminor radius $\sqrt{6}$ that lies on the plane 11x – 2y + 7z –35 = 0 (why?).